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# ON THE TOPOLOGICAL CHARACTERIZATION OF THE REAL LINE

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# On the topological characterization of the real line

by

A.E. Brouwer

Summary

In [1] Franklin and Krishnarao stated that a separable connected locally compact Hausdorff space in which each point is a strong cutpoint is homeomorphic to the real line. This is correct, but their proof is not. Here we give a proof of this and some related statements and two counter-examples.

## 1. Introduction

A connected space X is called treelike if for each pair of points p, q $\in$ X there is a third point r $\in$ X which separates p and q. Clearly a treelike space is Hausdorff.

A topological space is called rimcompact or,what is the same, (locally) peripherally compact if it has a base consisting of open sets with compact boundary.

A locally compact Hausdorff space is rimcompact, and a rimcompact Hausdorff space is completely regular.

A topological space is called weakly orderable if it can be ordered in such a way that the open orderintervals are open sets but do not necessarily constitute a base.

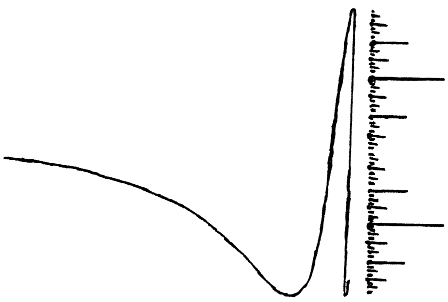
A point p of the connected space X is called a strong cutpoint (of X) if X\p decomposes into exactly two components.

In [1] Franklin and Krishnarao state

<u>Theorem 1</u> A separable connected locally compact Hausdorff space in which each point is a strong cutpoint is homeomorphic to the real line.

This is true, but their proof is incorrect as it would also apply to prove the same statement with locally compact replaced by rimcompact, which is incorrect as is shown by the separable metric counterexample:

$$X = \{(x,y) \in \mathbb{R}^2 \mid (x<0 \wedge y=\sin \tfrac{1}{x}) \vee \exists n\colon (0\leq x<2^{-n} \wedge \exists k\leq 2^{n-1}\colon |y| = \tfrac{2k-1}{2^n})\} \subset \mathbb{R}^2.$$



In fact they ascribe to Kok [3] the fancy-theorem: 'In a connected Hausdorff space each point being a strong cutpoint is equivalent to (S'): given three distinct points, someone separates the other two', against which he gives a counterexample.

Here theorem 1 will follow from

<u>Theorem 2</u> A separable connected locally compact Hausdorff space in which each point is a cutpoint is a treelike space and therefore by [2] locally connected and by [4] separable metric.

Without separability the space need not be orderable.

<u>Example</u>

Let $X = \{(x,y,z) \in \mathbb{R}^3 \mid z \geq 0\}$ with topology given by the local bases:

$$U_i(x,y,z) = \{x\} \times \{y\} \times (z-\tfrac{1}{i}, z+\tfrac{1}{i}) \qquad (z \geq \tfrac{1}{i})$$

$$U_{i,F}(x,y,0) = \{(u,v,w) \in X \mid (u+w-x)^2 + (y-v)^2 < \tfrac{1}{i^2}\} \setminus$$

$$\{(u,v,w) \in X \mid v = y \text{ and } x \neq u \in F\}$$

where $i \in \mathbb{N}$ and F is a finite set.
Then X is a locally compact connected Hausdorff space in which each point is a strong cutpoint, but not locally connected or orderable.

However, if not only the points but also the compact connected sets separate the space in exactly two pieces then the space is orderable:

<u>Theorem 3</u> A connected locally compact Hausdorff space X is orderable (without endpoints) if $X\backslash C$ consists of exactly two components for each compact connected subset C of X.

## 2. <u>The lemma</u>

Let X be a connected locally compact Hausdorff space in which each point is a cutpoint.
A subspace Y of X is called a brush in X if it contains a compact connected nondegenerated subset C (called the base of Y) such that:

1. if $p \in C$ then $C\backslash p$ is contained in one component $B_p$ of $X\backslash p$.

2. $Y = \bigcup_{p \in C} X\backslash B_p$.

<u>Lemma</u> Let X be a connected locally compact Hausdorff space in which each point is a cutpoint. If X is not treelike then there is a brush Y in X.

<u>Proof of the lemma</u>

Case A:
There is a point p such that a component S of $X\backslash p$ is not open. In this case, choose a point $q \in S \cap \overline{X\backslash S}$.
$\overline{X\backslash S}$ is a connected locally compact Hausdorff space, hence if V is a compact neighbourhood of q in $\overline{X\backslash S}$ not containing p then the component C of q in V must reach $\partial V$.
(Since in a connected space the component of a point in a compact neighbourhood V of that point must reach the boundary $\partial V$ of V.) But this component lies entirely in S and hence is the base for a brush. (If $r \in C$ then the component of $X\backslash r$ containing p also contains $X\backslash S$ and therefore $(\overline{X\backslash S})\backslash r$ and a fortiori $C\backslash r$).

Case B:

For each point $p \in X$ all components of $X\backslash p$ are open.

Since X is not treelike, it contains two points a, b which cannot be separated by a third point. Let for each point $p \in X$ $B_p$ be the component of $X\backslash p$ containing a or b. Let $S_p = X\backslash B_p$, then $S_p$ is closed and connected, and $S_p\backslash(S_p)^\circ = \{p\}$. Let $W_p = \cup\{S_q \mid p \in S_q\}$ then (since if $S_q \cap S_r \neq \emptyset$ then $S_q \subset S_r$ or $S_r \subset S_q$) if $W_p \cap W_q \neq \emptyset$ then $W_p = W_q$. Moreover, each $W_p$ is connected (since the $S_q$ are).

For each set $W_p$ there are two possibilities:

(i) it is open; this is the case if for each $r \in W_p$ there is a $q \neq r$ such that $r \in S_q$.

(ii) it contains exactly one non-interior point q; in this case $W_p = S_q$ and is therefore closed.

Assume first that some $W_p$ is open, then $a, b \notin W_p$.

Since X is connected and $W_p \neq X$ $W_p$ cannot be closed.

If $\overline{W_p}\backslash W_p = \{q\}$ then $p \in S_q$ so $q \in W_p$. A contradiction.

Therefore there are two points $q, r \in \overline{W_p}\backslash W_p$.

Now $\overline{W_p}$ is a locally compact connected subspace of X, so we can find two disjoint compact neighbourhoods $V_q$ and $V_r$ of q and r resp.

The components $C_q$ and $C_r$ of q and r in $V_q$ and $V_r$ (resp.) cannot both meet $W_p$, since if $q_1 \in C_q \cap W_p$ and $r_1 \in C_r \cap W_p$ then there is a point $s \in W_p$ such that $\{q_1,r_1\} \subset S_s$, and therefore s separates $q_1$ and $r_1$ from q and r. This however is impossible since s cannot lie both in $C_q$ and $C_r$.

Therefore we may suppose $C = C_q \subset \overline{W_p}\backslash W_p$. C is non-degenerated by the theorem already cited in case A, and therefore is the base of a brush. (By the same argument: if $t \in C$ then the component of $X\backslash t$ containing p or q also contains $\overline{W_p}\backslash t$ and a fortiori $C\backslash t$.)

Suppose now that each $W_p$ is closed, i.e. of the form $S_q$ for some q. Let $Z = \{q \mid W_q = S_q\}$. Z is closed, since $X\backslash Z = \bigcup_{q\in Z} S_q^\circ$ is open. Moreover $\{a,b\} \in Z$. If Z were connected we could find a non-degenerate compact connected subset C of Z and take $Y = \bigcup_{q\in C} S_q$ for our brush. (By the very definition of Z, $Z\backslash t$ is contained in one component of $X\backslash t$, sc. the component containing a or b.)

If $Z = Z_1 + Z_2$ then since X is connected either $\exists z_2 \in Z_2 : z_2 \in \overline{\bigcup_{q \in Z_1} S_q}$ or $\exists z_1 \in Z_1 : z_1 \in \overline{\bigcup_{q \in Z_2} S_q}$. Suppose $z_2 \in Z_2 \cap \overline{\bigcup_{q \in Z_1} S_q}$.
Let V be a compact neighbourhood of $z_2$ in the locally compact space $\overline{S}$ where $S = \bigcup_{q \in Z_1} S_q$ such that $V \cap Z_1 = \emptyset$.
Since V cannot contain a clopen neighbourhood of $z_2$ (each $S_q$ is connected) the component C of $z_2$ in V must reach $\partial V$. But this component cannot intersect S, hence $C \subset \overline{S} \setminus S \subset Z_2$, and again C is the base for the brush $Y = \bigcup_{q \in C} S_q$.
This proves the lemma.

## 3. Proof of the theorems

Suppose Y is a brush in X with base C.
Let for each $p \in C$ $B_p$ be the component of $X \setminus p$ containing $C \setminus p$.
Let $S_p = X \setminus B_p$. $S_p$ is connected and contains p.
If $p \neq q$, p, $q \in C$ then $q \in C \setminus p \subset B_p$ so $X \setminus B_q \subset B_p$ and $S_p \cap S_q = (X \setminus B_p) \cap (X \setminus B_q) = \emptyset$. Since X is regular, C contains at least $2^{\aleph_0}$ points p, and therefore X contains a collection of $2^{\aleph_0}$ pairwise disjoint open sets $\overset{\circ}{S}_p$. ($\overset{\circ}{S}_p$ is non-empty since p is a cutpoint of X.)
This implies that X cannot be separable nor satisfy the countable chain condition, which proves theorem 2.
Also $X \setminus C$ decomposes into at least $2^{\aleph_0}$ components, so the hypothesis of theorem 3 implies that X is treelike, and therefore, since in particular each point is strong cut point, that X is orderable (see [3] and [5]).
This proves theorem 3.
In the same way it follows from theorem 2 and the hypothesis of theorem 1 that X is orderable and therefore homeomorphic to $\mathbb{R}$. (A homeomorphism can be constructed in the usual way by first constructing an order-isomorphism between a countable dense subset of X and $\mathbb{Q}$, and then extending this to an order-isomorphism between X and $\mathbb{R}$. Since both have the order-topology, this is a homeomorphism.)
This proves theorem 1.

## 4. References

[1] S.P. Franklin and G.V. Krishnarao, "On the topological characterization of the real line: an addendum". J. London Math. Soc. (2) 3 (1971) 392.

[2] G.L. Gurin, "On treelike spaces". Vestnik Moskov. Univ. Ser. I. Mat. Meh. (1969), no. 1, 9-12. (Russian).

[3] H. Kok, "On conditions equivalent to the orderability of a connected space". Nieuw Archief voor Wiskunde (3), XVIII, 250-270 (1970).

[4] V.V. Proizvolov, "On peripherally bicompact treelike spaces". Soviet Math. Dokl. 10 (1969), no. 6, 1491-1493.

[5] E. Wattel, "An orderability theorem for connected spaces". Wiskundig Seminarium der Vrije Universiteit, De Boelelaan 1081, Amsterdam, Report no. 10 (May 1970).

Results quoted:

[2] A locally peripherally compact treelike space is locally connected.

[3] A treelike space in which each cutpoint is a strong cutpoint is weakly orderable.

[4] A separable locally peripherally compact treelike space is metrizable.

[5] A locally peripherally compact weakly orderable $T_1$ space is orderable.